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# Second Order Partial Differential Equations in Hilbert Spaces

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# Chapter 1

## Gaussian measures

This chapter is devoted to some basic results on Gaussian measures on separable Hilbert spaces, including the Cameron-Martin and Feldman-Hajek formulae. The greater part of the results are presented with complete proofs.

### 1.1 Introduction and preliminaries

We are given a real separable Hilbert space  $H$  (with norm  $|\cdot|$  and inner product  $\langle \cdot, \cdot \rangle$ ). The space of all linear bounded operators from  $H$  into  $H$ , equipped with the operator norm  $\|\cdot\|$ , will be denoted by  $L(H)$ . If  $T \in L(H)$ , then  $T^*$  is the adjoint of  $T$ . Moreover, by  $L^+(H)$  we shall denote the subset of  $L(H)$  consisting of all nonnegative symmetric operators. Finally, we shall denote by  $\mathcal{B}(H)$  the  $\sigma$ -algebra of all Borel subsets of  $H$ .

Before introducing Gaussian measures we need some results about trace class and Hilbert-Schmidt operators.

A linear bounded operator  $R \in L(H)$  is said to be of *trace class* if there exist two sequences  $(a_k), (b_k)$  in  $H$  such that

$$Ry = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k, \quad y \in H, \quad (1.1.1)$$

and

$$\sum_{k=1}^{\infty} |a_k| |b_k| < +\infty. \quad (1.1.2)$$

Notice that if (1.1.2) holds then the series in (1.1.1) is norm convergent. Moreover, it is not difficult to show that  $R$  is compact.



We shall denote by  $L_1(H)$  the set of all operators of  $L(H)$  of trace class.  $L_1(H)$ , endowed with the usual linear operations, is a Banach space with the norm

$$\|R\|_{L_1(H)} = \inf \left\{ \sum_{k=1}^{\infty} |a_k| |b_k| : Ry = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k, \ y \in H, \ (a_k), (b_k) \subset H \right\}.$$

We set  $L_1^+(H) = L^+(H) \cap L_1(H)$ . If an operator  $R$  is of trace class then its trace,  $\text{Tr } R$ , is defined by the formula

$$\text{Tr } R = \sum_{j=1}^{\infty} \langle R e_j, e_j \rangle,$$

where  $(e_j)$  is an orthonormal and complete basis on  $H$ . Notice that, if  $R$  is given by (1.1.1), we have

$$\text{Tr } R = \sum_{j=1}^{\infty} \langle a_j, b_j \rangle.$$

Thus the definition of the trace is independent on the choice of the basis and

$$|\text{Tr } R| \leq \|R\|_{L_1(H)}.$$

**Proposition 1.1.1** *Let  $S \in L_1(H)$  and  $T \in L(H)$ . Then*

(i)  $ST, TS \in L_1(H)$  and

$$\|TS\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|, \ \|ST\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|.$$

(ii)  $\text{Tr}(ST) = \text{Tr}(TS)$ .

**Proof.** (i) Assume that  $Sy = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k$ ,  $y \in H$ , where  $\sum_{k=1}^{\infty} |a_k| |b_k| < +\infty$ .

Then

$$STy = \sum_{k=1}^{\infty} \langle y, T^* a_k \rangle b_k, \ y \in H,$$

and

$$\sum_{k=1}^{\infty} |T^* a_k| |b_k| \leq \|T\| \sum_{k=1}^{\infty} |a_k| |b_k|.$$

It is therefore clear that  $ST \in L_1(H)$  and  $\|ST\|_{L_1(H)} \leq \|S\|_{L_1(H)}\|T\|$ . Similarly we can prove that  $\|TS\|_{L_1(H)} \leq \|S\|_{L_1(H)}\|T\|$ .

(ii) From part (i) it follows that

$$\text{Tr}(ST) = \sum_{k=1}^{\infty} \langle b_k, T^* a_k \rangle = \sum_{k=1}^{\infty} \langle T b_k, a_k \rangle.$$

In the same way  $\text{Tr}(TS) = \sum_{k=1}^{\infty} \langle a_k, T b_k \rangle$ , and the conclusion follows.  $\square$

We say that  $R \in L(H)$  is of Hilbert-Schmidt class if there exists an orthonormal and complete basis  $(e_k)$  in  $H$  such that

$$\sum_{k,j=1}^{\infty} |\langle S e_k, e_j \rangle|^2 < +\infty. \quad (1.1.3)$$

If (1.1.3) holds then we have

$$\sum_{k=1}^{\infty} |S e_k|^2 = \sum_{k,j=1}^{\infty} |\langle S e_k, e_j \rangle|^2 = \sum_{k,j=1}^{\infty} |\langle e_k, S^* e_j \rangle|^2 = \sum_{j=1}^{\infty} |S^* e_j|^2. \quad (1.1.4)$$

Now if  $(f_k)$  is another complete orthonormal basis in  $H$ , we have

$$\sum_{m=1}^{\infty} |S f_m|^2 = \sum_{m,n=1}^{\infty} |\langle S f_m, e_n \rangle|^2 = \sum_{m,n=1}^{\infty} |\langle f_m, S^* e_n \rangle|^2 = \sum_{n=1}^{\infty} |S^* e_n|^2.$$

Thus, by (1.1.4) we see that the assertion (1.1.3) is independent of the choice of the complete orthonormal basis  $(e_k)$ . We shall denote by  $L_2(H)$  the space of all Hilbert-Schmidt operators on  $H$ .  $L_2(H)$ , endowed with the norm

$$\|S\|_{L_2(H)}^2 = \sum_{k,j=1}^{\infty} |\langle S e_k, e_j \rangle|^2 = \sum_{k=1}^{\infty} |S e_k|^2,$$

is a Banach space.

**Proposition 1.1.2** *Let  $S, T \in L_2(H)$ . Then  $ST \in L_1(H)$  and*

$$\|ST\|_{L_1(H)} \leq \|S\|_{L_2(H)} \|T\|_{L_2(H)}. \quad (1.1.5)$$

**Proof.** Let  $(e_k)$  be a complete and orthonormal basis in  $H$ , then

$$\begin{aligned} Ty &= \sum_{k=1}^{\infty} \langle Ty, e_k \rangle e_k = \sum_{k=1}^{\infty} \langle y, T^* e_k \rangle e_k, \\ STy &= \sum_{k=1}^{\infty} \langle y, T^* e_k \rangle S e_k. \end{aligned}$$

Consequently  $ST \in L_1(H)$  and

$$\begin{aligned} \|ST\|_{L_1(H)} &\leq \sum_{k=1}^{\infty} |T^* e_k| |S e_k| \leq \left( \sum_{k=1}^{\infty} |T^* e_k|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} |S e_k|^2 \right)^{1/2} \\ &= \|T\|_{L_2(H)} \|S\|_{L_2(H)}. \end{aligned}$$

Therefore the conclusion follows.  $\square$

**Warning.** If  $S$  and  $T$  are bounded operators, and  $ST$  is of trace class then in general  $TS$  is not, as the following example, provided by S. Peszat [183], shows.

Define two linear operators  $S$  and  $T$  on the product space  $H \times H$ , by

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$ST = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad TS = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

and it is enough to take  $B$  of trace class and  $A$  not of trace class.  $\square$

We have also the following result, see e.g. A. Pietsch [187].

**Proposition 1.1.3** *Assume that  $S$  is a compact self-adjoint operator, and that  $(\lambda_k)$  are its eigenvalues (repeated according to their multiplicity).*

(i)  $S \in L_1(H)$  if and only if  $\sum_{k=1}^{\infty} |\lambda_k| < +\infty$ . Moreover  $\|S\|_{L_1(H)} = \sum_{k=1}^{\infty} |\lambda_k|$ ,

and  $\text{Tr } S = \sum_{k=1}^{\infty} \lambda_k$ .

(ii)  $S \in L_2(H)$  if and only if  $\sum_{k=1}^{\infty} |\lambda_k|^2 < +\infty$ . Moreover

$$\|S\|_{L_2(H)} = \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{1/2}.$$

More generally let  $S$  be a compact operator on  $H$ . Denote by  $(\lambda_k)$  the sequence of all positive eigenvalues of the operator  $(S^*S)^{1/2}$ , repeated according to their multiplicity. Denote by  $L_p(H)$ ,  $p > 0$ , the set of all operators  $S$  such that

$$\|S\|_{L_p(H)} = \left( \sum_{k=1}^{\infty} \lambda_k^p \right)^{1/p} < +\infty. \quad (1.1.6)$$

Operators belonging to  $L_1(H)$  and  $L_2(H)$  are precisely the trace class and the Hilbert-Schmidt operators.

The following result holds, see N. Dunford and J. T. Schwartz [107].

**Proposition 1.1.4** *Let  $S \in L_p(H)$ ,  $T \in L_q(H)$  with  $p > 0, q > 0$ . Then  $ST \in L_r(H)$  with  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ , and*

$$\|TS\|_{L_r(H)} \leq 2^{1/r} \|S\|_{L_p(H)} \|T\|_{L_q(H)}. \quad (1.1.7)$$

## 1.2 Definition and first properties of Gaussian measures

### 1.2.1 Measures in metric spaces

If  $E$  is a metric space, then  $\mathcal{B}(E)$  will denote the Borel  $\sigma$ -algebra, that is the smallest  $\sigma$ -algebra of subsets of  $E$  which contains all closed (open) subsets of  $E$ .

Let metric spaces  $E_1, E_2$  be equipped with  $\sigma$ -fields  $\mathcal{E}_1, \mathcal{E}_2$  respectively. Measurable mappings  $X : E_1 \rightarrow E_2$  will often be called *random variables*. If  $\mu$  is a measure on  $(E_1, \mathcal{E}_1)$ , then its image by the transformation  $X$  will be denoted by  $X \circ \mu$  :

$$X \circ \mu(A) = \mu(X^{-1}(A)), \quad A \in \mathcal{E}_2.$$

We call  $X \circ \mu$  the *law* or the *distribution* of  $X$ , and we set  $X \circ \mu = \mathcal{L}(X)$ .

If  $\nu$  and  $\mu$  are two finite measures on  $(E, \mathcal{E})$  such that  $\Gamma \in \mathcal{E}$ ,  $\mu(\Gamma) = 0$  implies  $\nu(\Gamma) = 0$  then one writes  $\nu \ll \mu$  and one says that  $\nu$  is *absolutely continuous* with respect to  $\mu$ . If there exist  $A, B \in \mathcal{E}$  such that  $A \cap B = \emptyset$ ,  $\mu(A) = \nu(B) = 1$ , one says that  $\mu$  and  $\nu$  are *singular*.

If  $\nu \ll \mu$  then by the Radon-Nikodým theorem there exists  $g \in L^1(E, \mathcal{E}, \mu)$  nonnegative such that

$$\nu(\Gamma) = \int_{\Gamma} g(x) \mu(dx), \quad \Gamma \in \mathcal{E}.$$

The function  $g$  is denoted by  $\frac{d\nu}{d\mu}$ .

If  $\nu \ll \mu$  and  $\mu \ll \nu$  then one says that  $\mu$  and  $\nu$  are *equivalent* and writes  $\mu \sim \nu$ .

We have the following change of variable formula. If  $\varphi$  is a nonnegative measurable real function on  $E_2$ , then

$$\int_{E_1} \varphi(X(x)) \mu(dx) = \int_{E_2} \varphi(y) X \circ \mu(dy). \quad (1.2.1)$$

Let  $\mu$  and  $\nu$  be two measures on a separable Hilbert space  $H$ ; if  $T \circ \mu = T \circ \nu$  for any linear operator  $T : H \rightarrow \mathbb{R}^n$ ,  $n \in \mathbb{N}$ , then  $\mu = \nu$ .

Random variables  $X_1, \dots, X_n$  are said to be *independent* if

$$\mathcal{L}(X_1, \dots, X_n) = \mathcal{L}(X_1) \times \dots \times \mathcal{L}(X_n).$$

A family of random variables  $(X_\alpha)_{\alpha \in A}$  is said to be independent, if any finite subset of the family is independent.

Probability measures on a separable Hilbert space  $H$  will always be regarded as defined on  $\mathcal{B}(H)$ . If  $\mu$  is a probability measure on  $H$ , then its Fourier transform is defined by

$$\hat{\mu}(\lambda) = \int_H e^{i\langle \lambda, x \rangle} \mu(dx), \quad \lambda \in H;$$

$\hat{\mu}$  is called the *characteristic function* of  $\mu$ . One can show that if the characteristic functions of two measures are identical, then the measures are identical as well.

### 1.2.2 Gaussian measures

We first define Gaussian measures on  $\mathbb{R}$ . If  $a \in \mathbb{R}$  we set

$$N_{a,0}(dx) = \delta_a(dx),$$

where  $\delta_a$  is the Dirac measure at  $a$ . If moreover  $\lambda > 0$  we set

$$N_{a,\lambda}(dx) = \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-a)^2}{2\lambda}} dx.$$

The Fourier transform of  $N_{a,\lambda}$  is given by

$$\widehat{N_{a,\lambda}}(h) = \int_{\mathbb{R}} e^{ihx} N_{a,\lambda}(dx) = e^{iah - \frac{1}{2}\lambda h^2}, \quad h \in \mathbb{R}.$$

More generally we show now that in an arbitrary separable Hilbert space and for arbitrary  $Q \in L_1^+(H)$  there exists a unique measure  $N_{a,Q}$  such that

$$\widehat{N_{a,Q}}(h) = \int_H e^{i\langle h,x \rangle} N_{a,Q}(dx) = e^{i\langle h,a \rangle - \frac{1}{2}\langle Qh,h \rangle}, \quad h \in H.$$

Let in fact  $Q \in L_1^+(H)$ . Then there exist a complete orthonormal system  $(e_k)$  on  $H$  and a sequence of nonnegative numbers  $(\lambda_k)$  such that  $Qe_k = \lambda_k e_k$ ,  $k \in \mathbb{N}$ . We set  $x_h = \langle x, e_h \rangle$ ,  $h \in \mathbb{N}$ , and  $P_n x = \sum_{k=1}^n x_k e_k$ ,  $x \in H$ ,  $n \in \mathbb{N}$ . Let us introduce an isomorphism  $\gamma$  from  $H$  into  $\ell^2$ :<sup>(1)</sup>

$$x \in H \rightarrow \gamma(x) = (x_k) \in \ell^2.$$

In the following we shall always identify  $H$  with  $\ell^2$ . In particular we shall write  $P_n x = (x_1, \dots, x_n)$ ,  $x \in \ell^2$ .

A subset  $I$  of  $H$  of the form  $I = \{x \in H : (x_1, \dots, x_n) \in B\}$ , where  $B \in \mathcal{B}(\mathbb{R}^n)$ , is said to be *cylindrical*. It is easy to see that the  $\sigma$ -algebra generated by all cylindrical subsets of  $H$  coincides with  $\mathcal{B}(H)$ .

**Theorem 1.2.1** *Let  $a \in H$ ,  $Q \in L_1^+(H)$ . Then there exists a unique probability measure  $\mu$  on  $(H, \mathcal{B}(H))$  such that*

$$\int_H e^{i\langle h,x \rangle} \mu(dx) = e^{i\langle a,h \rangle - \frac{1}{2}\langle Qh,h \rangle}, \quad h \in H. \quad (1.2.2)$$

Moreover  $\mu$  is the restriction to  $H$  (identified with  $\ell^2$ ) of the product measure

$$\bigotimes_{k=1}^{\infty} \mu_k = \bigotimes_{k=1}^{\infty} N_{a_k, \lambda_k},$$

defined on  $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ .<sup>(2)</sup>

We set  $\mu = N_{a,Q}$ , and call  $a$  the *mean* and  $Q$  the *covariance operator* of  $\mu$ . Moreover  $N_{0,Q}$  will be denoted by  $N_Q$ .

**Proof of Theorem 1.2.1.** Since a characteristic function uniquely determines the measure, we have only to prove existence.

Let us consider the sequence of Gaussian measures  $(\mu_k)$  on  $\mathbb{R}$  defined as  $\mu_k = N_{a_k, \lambda_k}$ ,  $k \in \mathbb{N}$ , and the product measure  $\mu = \bigotimes_{k=1}^{\infty} \mu_k$  in  $\mathbb{R}^\infty$ , see e.g.

<sup>1</sup>For any  $p \geq 1$ , we denote by  $\ell^p$  the Banach space of all sequences  $(x_k)$  of real numbers such that  $|x|_p := (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} < +\infty$ .

<sup>2</sup>We shall consider  $\mathbb{R}^\infty$  as a metric space with the distance  $d(x, y) := \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$ ,  $x, y \in \mathbb{R}^\infty$

P. R. Halmos [141, §38.B]. We want to prove that  $\mu$  is concentrated on  $\ell^2$ , (that it is clearly a Borel subset of  $\mathbb{R}^\infty$ ). For this it is enough to show that

$$\int_{\ell^\infty} |x|_{\ell^2}^2 \mu(dx) < +\infty. \quad (1.2.3)$$

We have in fact, by the monotone convergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^\infty} |x|_{\ell^2}^2 \mu(dx) &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^\infty} x_k^2 \mu(dx) = \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}} (x_k - a_k)^2 \mu_k(dx) + a_k^2 \right) \\ &= \sum_{k=1}^{\infty} (\lambda_k + a_k^2) = \text{Tr } Q + |a|^2 < +\infty. \end{aligned}$$

Now we consider the restriction of  $\mu$  to  $\ell^2$ , which we still denote by  $\mu$ . We have to prove that (1.2.2) holds. Setting  $\nu_n = \prod_{k=1}^n \mu_k$ , we have

$$\begin{aligned} \int_{\ell^2} e^{i\langle x, h \rangle} \mu(dx) &= \lim_{n \rightarrow \infty} \int_{\ell^2} e^{i\langle P_n h, P_n x \rangle} \mu(dx) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} e^{i\langle P_n h, P_n x \rangle} \nu_n(dx) = \lim_{n \rightarrow \infty} e^{i\langle P_n h, P_n a \rangle - \frac{1}{2} \langle Q P_n h, P_n h \rangle} \\ &= e^{i\langle h, a \rangle - \frac{1}{2} \langle Q h, h \rangle}. \quad \square \end{aligned}$$

If the law of a random variable is a Gaussian measure, then the random variable is called *Gaussian*. It easily follows from Theorem 1.2.1 that a random variable  $X$  with values in  $H$  is Gaussian if and only if for any  $h \in H$  the real valued random variable  $\langle h, X \rangle$  is Gaussian.

**Remark 1.2.2** From the proof of Theorem 1.2.1 it follows that

$$\int_H |x|^2 N_{a,Q}(dx) = \text{Tr } Q + |a|^2. \quad (1.2.4)$$

**Proposition 1.2.3** *Let  $T \in L(H)$ , and  $a \in H$ , and let  $\Gamma x = Tx + a$ ,  $x \in H$ . Then  $\Gamma \circ N_{m,Q} = N_{Tm+a, TQT^*}$ .*

**Proof.** Notice that, by the change of variables formula (1.2.1), we have

$$\begin{aligned} \int_H e^{i\langle \lambda, y \rangle} \Gamma \circ N_{m,Q}(dy) &= \int_H e^{i\langle \lambda, \Gamma x \rangle} N_{m,Q}(dy) \\ &= \int_H e^{i\langle \lambda, Tx+a \rangle} N_{m,Q}(dy) = e^{i\langle \lambda, a \rangle} e^{i\langle T^* \lambda, m \rangle - \frac{1}{2} \langle QT^* \lambda, T^* \lambda \rangle}. \end{aligned}$$

This shows the result.  $\square$

### 1.2.3 Computation of some Gaussian integrals

We are here given a Gaussian measure  $N_{a,Q}$ . We set

$$L^2(H, N_{a,Q}) = L^2(H, \mathcal{B}(H), N_{a,Q}).$$

The following identities can be easily proved, using (1.2.2).

**Proposition 1.2.4** *We have*

$$\int_H x N_{a,Q}(dx) = a, \quad (1.2.5)$$

$$\int_H \langle x - a, y \rangle \langle x - a, z \rangle N_{a,Q}(dx) = \langle Qy, z \rangle. \quad (1.2.6)$$

$$\int_H |x - a|^2 N_{a,Q}(dx) = \text{Tr } Q. \quad (1.2.7)$$

**Proof.** We prove as instance (1.2.6). We have

$$\int_H x N_{a,Q}(dx) = \lim_{n \rightarrow \infty} \int_H P_n x N_{a,Q}(dx).$$

But

$$\int_H P_n x N_{a,Q}(dx) = (2\pi)^{-n/2} \prod_{k=1}^n \int_{\mathbb{R}} x_k \lambda_k^{-1/2} e^{-\frac{(x_k - a_k)^2}{2\lambda_k}} dx_k = a_k,$$

and the conclusion follows.  $\square$

**Proposition 1.2.5** *For any  $h \in H$ , the exponential function  $E_h$ , defined as*

$$E_h(x) = e^{\langle h, x \rangle}, \quad x \in H,$$

*belongs to  $L^p(H, N_{a,Q})$ ,  $p \geq 1$ , and*

$$\int_H e^{\langle h, x \rangle} N_{a,Q}(dx) = e^{\langle a, h \rangle} e^{\frac{1}{2} \langle Qh, h \rangle}. \quad (1.2.8)$$

*Moreover the subspace of  $L^2(H, N_{a,Q})$  spanned by all  $E_h$ ,  $h \in H$ , is dense on  $L^2(H, N_{a,Q})$ .*

**Proof.** We have

$$\int_H e^{\langle P_n h, P_n x \rangle} N_{a,Q}(dx) = e^{\langle P_n a, P_n h \rangle} e^{\frac{1}{2} \langle Q P_n h, P_n h \rangle}.$$



Letting  $n$  tend to 0 this gives (1.2.8).

Let us prove the last statement. Let  $\varphi \in L^2(H, N_{a,Q})$  be such that

$$\int_H e^{\langle h, x \rangle} \varphi(x) N_{a,Q}(dx) = 0, \quad h \in H.$$

Denote by  $\varphi^+$  and  $\varphi^-$  the positive and negative parts of  $\varphi$ . Then

$$\int_H e^{\langle h, x \rangle} \varphi^+(x) N_{a,Q}(dx) = \int_H e^{\langle h, x \rangle} \varphi^-(x) N_{a,Q}(dx), \quad h \in H.$$

Let us define two measures

$$\mu(dx) = \varphi^+(x) N_{a,Q}(dx), \quad \nu(dx) = \varphi^-(x) N_{a,Q}(dx).$$

Then  $\mu$  and  $\nu$  are finite measures such that

$$\int_H e^{\langle h, x \rangle} \mu(dx) = \int_H e^{\langle h, x \rangle} \nu(dx), \quad h \in H.$$

Let  $T$  be any linear transformation from  $H$  into  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ . Then for any  $\lambda \in \mathbb{R}^n$

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\langle \lambda, z \rangle} T \circ \mu(dz) &= \int_H e^{\langle \lambda, Tx \rangle} \mu(dx) = \int_H e^{\langle T^* \lambda, x \rangle} \mu(dx) \\ &= \int_H e^{\langle T^* \lambda, x \rangle} \nu(dx) = \int_{\mathbb{R}^n} e^{\langle \lambda, z \rangle} T \circ \nu(dz). \end{aligned}$$

By a well known finite dimensional result  $T \circ \mu = T \circ \nu$ . Consequently measures  $\mu$  and  $\nu$  are identical and so  $\varphi = 0$ .  $\square$

### 1.2.4 The reproducing kernel

Here we are given an operator  $Q \in L_1^+(H)$ . We denote as before by  $(e_k)$  a complete orthonormal system in  $H$  and by  $(\lambda_k)$  a sequence of positive numbers such that  $Qe_k = \lambda_k e_k$ ,  $k \in \mathbb{N}$ .

The subspace  $Q^{1/2}(H)$  is called the *reproducing kernel* of the measure  $N_Q$ . If  $\text{Ker } Q = \{0\}$ ,  $Q^{1/2}(H)$  is dense on  $H$ . In fact, if  $x_0 \in H$  is such that  $\langle Q^{1/2}h, x_0 \rangle = 0$  for all  $h \in H$ , we have  $Q^{1/2}x_0 = 0$  and so  $Qx_0 = 0$ , which yields  $x_0 = 0$ .

Let  $\text{Ker } Q = \{0\}$ . We are now going to introduce an isomorphism  $W$  from  $H$  into  $L^2(H, N_Q)$  that will play an important rôle in the following. The isomorphism  $W$  is defined by

$$f \in Q^{1/2}(H) \rightarrow W_f \in L^2(H, N_Q), \quad W_f(x) = \langle Q^{-1/2}f, x \rangle, \quad x \in H.$$

By (1.2.7) it follows that

$$\int_H W_f(x) W_g(x) N_Q(dx) = \langle f, g \rangle, \quad f, g \in H.$$

Thus  $W$  is an isometry and it can be uniquely extended to all of  $H$ . It will be denoted by the same symbol. For any  $f \in H$ ,  $W_f$  is a real Gaussian random variable  $N_{|f|^2}$ .

More generally, for arbitrary elements  $f_1, \dots, f_n$ ,  $(W_{f_1}, \dots, W_{f_n})$  is a Gaussian vector with mean 0 and covariance matrix  $(\langle f_i, f_j \rangle)$ . If  $\text{Ker } Q \neq \{0\}$  then the transformation  $f \rightarrow W_f$  can be defined in exactly the same way but only for  $f \in H_0 = \overline{Q^{1/2}(H)}$ . We will write in some cases  $\langle Q^{-1/2}y, f \rangle$  instead of  $W_f(y)$ .

The proof of the following proposition is left as an exercise to the reader.

**Proposition 1.2.6** *For any orthonormal sequence  $(f_n)$  in  $H$ , the family*

$$1, W_{f_n}, W_{f_k} W_{f_l}, 2^{-1/2} (W_{f_m}^2 - 1), \quad m, n, k, l \in \mathbb{N}, \quad k \neq l,$$

*is orthonormal in  $L^2(H, N_Q)$ .*

Next we consider the function  $f \rightarrow e^{W_f}$ .

**Proposition 1.2.7** *The transformation  $f \rightarrow e^{W_f}$  acts continuously from  $H$  into  $L^2(H, N_Q)$ , and*

$$\begin{aligned} \int_H e^{W_f(x)} N_Q(dx) &= e^{\frac{1}{2}|f|^2}, \\ \int_H e^{i\lambda W_f(x)} N_Q(dx) &= e^{-\frac{1}{2}\lambda^2|f|^2}, \quad \lambda \in \mathbb{R}. \end{aligned} \tag{1.2.9}$$

**Proof.** Since  $W_f$  is Gaussian with law  $N_{0,|f|^2}$ , (1.2.9) follows. Moreover, taking into account (1.2.8) it follows that

$$\begin{aligned} \int_H [e^{W_f} - e^{W_g}]^2 dN_Q &= \int_H [e^{2W_f} - 2e^{W_f+W_g} + e^{2W_g}] dN_Q \\ &= e^{2|f|^2} - 2e^{\frac{1}{2}|f+g|^2} + e^{2|g|^2} = [e^{|f|^2} - e^{|g|^2}]^2 + 2e^{|f|^2+|g|^2} [1 - e^{-\frac{1}{2}|f-g|^2}], \end{aligned}$$

which shows that  $W_f$  is locally uniformly continuous on  $H$ .  $\square$

Let us define the determinant of  $1 + S$  where  $S$  is a compact self-adjoint operator in  $L_1(H)$  :

$$\det(1 + S) = \prod_{k=1}^{\infty} (1 + s_k),$$

where  $(s_k)$  is the sequence of eigenvalues of  $S$  (repeated according to their multiplicity).

**Proposition 1.2.8** *Assume that  $M$  is a symmetric operator such that  $Q^{1/2}MQ^{1/2} < 1$ , <sup>(3)</sup> and let  $b \in H$ . Then*

$$\begin{aligned} & \int_H \exp \left\{ \frac{1}{2} \langle My, y \rangle + \langle b, y \rangle \right\} N_Q(dy) \\ &= \left[ \det(1 - Q^{1/2}MQ^{1/2}) \right]^{-1/2} \exp \left\{ \frac{1}{2} |(1 - Q^{1/2}MQ^{1/2})^{-1/2} Q^{1/2}b|^2 \right\}. \end{aligned} \quad (1.2.10)$$

**Proof.** Let  $(g_n)$  be an orthonormal basis for the operator  $Q^{1/2}MQ^{1/2}$ , and let  $(\gamma_n)$  be the sequence of the corresponding eigenvalues.

**Claim 1.** We have

$$\langle b, x \rangle = \sum_{k=1}^{\infty} \langle Q^{1/2}b, g_n \rangle W_{g_n}(x), \quad N_Q\text{-a.e.}$$

**Claim 2.** We have

$$\langle Mx, x \rangle = \sum_{n=1}^{\infty} \gamma_n |W_{g_n}(x)|^2, \quad N_Q\text{-a.e.},$$

the series being convergent in  $L^1(H, N_Q)$ .

We shall only prove the more difficult second claim.

Let  $P_N = \sum_{k=1}^N e_k \otimes e_k$ . <sup>(4)</sup> Then for any  $x \in H$  we have

$$\begin{aligned} \langle MP_Nx, P_Nx \rangle &= \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_Nx, Q^{-1/2}P_Nx \rangle \\ &= \sum_{n=1}^{\infty} \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_Nx, g_n \rangle \langle Q^{-1/2}P_Nx, g_n \rangle \\ &= \sum_{n=1}^{\infty} \gamma_n |\langle Q^{-1/2}P_Nx, g_n \rangle|^2. \end{aligned}$$

Consequently, for each fixed  $x$

$$\langle MP_Nx, P_Nx \rangle = \sum_{n=1}^{\infty} \gamma_n |W_{P_Ng_n}|^2, \quad N \in \mathbb{N}.$$

<sup>3</sup>This means that  $\langle Q^{1/2}MQ^{1/2}x, x \rangle < |x|^2$  for any  $x \in H$  different from 0.

<sup>4</sup>We remember that  $(e_k)$  is the sequence of eigenvectors of  $Q$ .

Moreover for each  $L \in \mathbb{N}$

$$\begin{aligned}
& \int_H \left| \langle MP_N x, P_N x \rangle - \sum_{n=1}^L \gamma_n |W_{P_N g_n}|^2 \right| N_Q(dx) \\
& \leq \sum_{n=L+1}^{\infty} |\gamma_n| \int_H |W_{P_N g_n}|^2 N_Q(dx) \\
& = \sum_{n=L+1}^{\infty} |\gamma_n| |P_N g_n|^2 \leq \sum_{n=L+1}^{\infty} |\gamma_n|.
\end{aligned}$$

As  $N \rightarrow \infty$  then  $P_N x \rightarrow x$  and  $W_{P_N g_n} \rightarrow W_{g_n}$  in  $L^2(H, N_Q)$ . Passing to subsequences if needed, and using the Fatou lemma, we see that

$$\int_H \left| \langle Mx, x \rangle - \sum_{n=1}^L \gamma_n |W_{g_n}|^2 \right| N_Q(dx) \leq \sum_{n=L+1}^{\infty} |\gamma_n|.$$

Therefore the claim is proved.

By the claims it follows that

$$\begin{aligned}
& \exp \left\{ \frac{1}{2} \langle Mx, x \rangle + \langle b, x \rangle \right\} \\
& = \lim_{L \rightarrow \infty} \exp \left\{ \sum_{n=1}^L \frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2} b, g_n \rangle W_{g_n}(x) \right\},
\end{aligned}$$

with a.e. convergence with respect to  $N_Q$  for a suitable subsequence. Using the fact that  $(W_{g_n})$  are independent Gaussian random variables, we obtain, by a direct calculation, for  $p \geq 1$ ,

$$\begin{aligned}
& \int_H \exp \left\{ p \sum_{n=1}^L \frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + p \langle Q^{1/2} b, g_n \rangle W_{g_n}(x) \right\} N_Q(dx) \\
& = \left[ \prod_{n=1}^L (1 - p\gamma_n) \right]^{-1/2} \exp \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2} b, g_n \rangle|^2}{1 - p\gamma_n} \right\}.
\end{aligned}$$

Since  $\gamma_n < 1$ , and  $\sum_{n=1}^{\infty} |\gamma_n| < \infty$ , there exists  $p > 1$  such that  $p\gamma_n < 1$ , for all  $n \in \mathbb{N}$ . Therefore

$$\begin{aligned} & \lim_{L \rightarrow \infty} \prod_{n=1}^L (1 - p\gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - p\gamma_n} \right\} \\ &= \left[ \prod_{n=1}^{\infty} (1 - p\gamma_n) \right]^{-1/2} \exp \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - p\gamma_n} \right\}. \end{aligned}$$

So the sequence  $\left( \exp \left\{ \sum_{n=1}^L \left[ \frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n \rangle W_{g_n}(x) \right] \right\} \right)$  is uniformly integrable. Consequently, passing to the limit, we find

$$\begin{aligned} & \int_H \exp \{ 1/2 \langle My, y \rangle + \langle b, y \rangle \} N_Q(dy) \\ &= \lim_{L \rightarrow \infty} \int_H \exp \left\{ \sum_{n=1}^L \left[ 1/2 \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n \rangle W_{g_n}(x) \right] \right\} N_Q(dx) \\ &= \lim_{L \rightarrow \infty} \prod_{n=1}^L (1 - \gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - \gamma_n} \right\} \\ &= \prod_{n=1}^{\infty} (1 - \gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - \gamma_n} \right\} \\ &= \left( \det(1 - Q^{1/2}MQ^{1/2}) \right)^{-1/2} \exp \left\{ \frac{1}{2} |(1 - Q^{1/2}MQ^{1/2})^{-1/2} Q^{1/2}b|^2 \right\}. \quad \square \end{aligned}$$

**Remark 1.2.9** It follows from the proof of the proposition that

$$\langle Mx, x \rangle = \sum_{k=1}^{\infty} \gamma_k W_{g_k}^2(x) = \sqrt{2} \sum_{k=1}^{\infty} \gamma_k \left[ 2^{-1/2} (W_{g_k}^2(x) - 1) \right] + \sum_{k=1}^{\infty} \gamma_k,$$

and so, by Proposition 1.2.6, we have

$$\begin{aligned}
\int_H [\langle Mx, x \rangle]^2 N_Q(dx) &= 2 \sum_{k=1}^{\infty} \gamma_n^2 + \left( \sum_{k=1}^{\infty} \gamma_n \right)^2 \\
&= 2 \|Q^{1/2} M Q^{1/2}\|_{L_2(H)}^2 + (\text{Tr } Q^{1/2} M Q^{1/2})^2 \\
&< +\infty.
\end{aligned}$$

**Proposition 1.2.10** *Let  $T \in L_1(H)$ . Then there exists the limit*

$$\langle TQ^{-1/2}y, Q^{-1/2}y \rangle := \lim_{n \rightarrow \infty} \langle TQ^{-1/2}P_n y, Q^{-1/2}P_n y \rangle, \text{ } N_Q\text{-a.e.,}$$

where  $P_n = \sum_{k=1}^n e_k \otimes e_k$ .

Moreover we have the following expansion in  $L^2(H, N_Q)$ :

$$\begin{aligned}
\langle TQ^{-1/2}y, Q^{-1/2}y \rangle &= \sum_{n=1}^{\infty} \langle Tg_n, g_n \rangle + \sum_{m \neq n=1}^{\infty} \langle Tg_n, g_m \rangle W_{g_n} W_{g_m} \\
&\quad \times \sqrt{2} \sum_{n=1}^{\infty} \langle Tg_n, g_n \rangle \left[ 2^{-1/2} (W_{g_n}^2 - 1) \right]. \quad (1.2.11)
\end{aligned}$$

The proof of the following result is similar to that of Claim 2 in the proof of Proposition 1.2.8 and it is left to the reader.

**Proposition 1.2.11** *Assume that  $M$  is a symmetric trace-class operator such that  $M < 1$ ,<sup>(5)</sup> and  $b \in H$ . Then*

$$\begin{aligned}
\int_H \exp \left\{ 1/2 \langle MQ^{-1/2}y, Q^{-1/2}y \rangle + \langle b, Q^{-1/2}y \rangle \right\} N_Q(dy) \\
= (\det(1 - M))^{-1/2} e^{\frac{1}{2} |(1-M)^{-1/2}b|^2}. \quad (1.2.12)
\end{aligned}$$

### 1.3 Absolute continuity of Gaussian measures

We consider here two Gaussian measures  $\mu, \nu$ . We want to prove the Feldman-Hajek theorem, that is they are either singular or equivalent.

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<sup>5</sup>That is  $\langle Mx, x \rangle < |x|^2$  for all  $x \neq 0$ .

In §1.3.1 we recall some results on equivalence of measures on  $\mathbb{R}^\infty$  including the Kakutani theorem. In §1.3.2 we consider the case when  $\mu = N_Q$  and  $\nu = N_{a,Q}$  with  $Q \in L_1^+(H)$  and  $a \in H$ , proving the Cameron-Martin formula. Finally in §1.3.3 we consider the more difficult case when  $\mu = N_Q$  and  $\nu = N_R$  with  $Q, R \in L_1^+(H)$ .

### 1.3.1 Equivalence of product measures in $\mathbb{R}^\infty$

It is convenient to introduce the notion of *Hellinger* integral.

Let  $\mu, \nu$  be probability measures on a measurable space  $(E, \mathcal{E})$ . Then  $\lambda = \frac{1}{2}(\mu + \nu)$  is also a probability measure on  $(E, \mathcal{E})$  and we have obviously

$$\mu \ll \lambda, \quad \nu \ll \lambda.$$

We define the *Hellinger integral* by

$$H(\mu, \nu) = \int_E \left[ \frac{d\mu}{d\lambda}(x) \frac{d\nu}{d\lambda}(x) \right]^{1/2} \lambda(dx).$$

Instead of  $\frac{1}{2}(\mu + \nu)$  one could choose as  $\lambda$  any measure equivalent to  $\frac{1}{2}(\mu + \nu)$  without changing the value of  $H(\mu, \nu)$ .

By using Hölder's inequality we see that

$$|H(\mu, \nu)|^2 \leq \int_E \frac{d\mu}{d\lambda}(x) \lambda(dx) \int_E \frac{d\nu}{d\lambda}(x) \lambda(dx) = 1,$$

so that  $0 \leq H(\mu, \nu) \leq 1$ .

**Exercise 1.3.1** (a) Let  $\mu = N_q$  and  $\nu = N_{a,q}$ , where  $a \in \mathbb{R}$  and  $q > 0$ . Show that we have

$$H(\mu, \nu) = e^{-\frac{a^2}{4q}}. \quad (1.3.1)$$

(b) Let  $\mu = N_q$  and  $\nu = N_\rho$ , where  $q, \rho > 0$ . Show that we have

$$H(\mu, \nu) = \left[ \frac{4q\rho}{(q + \rho)^2} \right]^{1/4}. \quad (1.3.2)$$

**Proposition 1.3.2** Assume that  $H(\mu, \nu) = 0$ . Then the measures  $\mu$  and  $\nu$  are singular.

**Proof.** Set  $\alpha = \frac{d\mu}{d\lambda}$ ,  $\beta = \frac{d\nu}{d\lambda}$ . Since  $H(\mu, \nu) = \int_{\Omega} \sqrt{\alpha\beta} d\lambda = 0$ , we have  $\alpha\beta = 0$ ,  $\lambda$ -a.e. Consequently, setting

$$A = \{\omega \in \Omega : \alpha(\omega) = 0\}, \quad B = \{\omega \in \Omega : \beta(\omega) = 0\},$$

we have  $\lambda(A \cup B) = 1$ . This means that  $\lambda(C) = 0$  where  $C = \Omega \setminus (A \cup B)$ , and hence  $\mu(C) = \nu(C) = 0$ . Then, as

$$\mu(A) = \int_A \alpha d\lambda = 0, \quad \nu(B) = \int_B \beta d\lambda = 0,$$

we have that  $\mu$  and  $\nu$  are singular since

$$\mu(A \cup C) = \nu(B) = 0, \quad (A \cup C) \cap B = \emptyset. \quad \square$$

**Proposition 1.3.3** *Let  $\mathcal{G} \subset \mathcal{E}$  be a  $\sigma$ -algebra, and let  $\mu_{\mathcal{G}}$  and  $\nu_{\mathcal{G}}$  be the restrictions of  $\mu$  and  $\nu$  to  $(E, \mathcal{G})$ . Then we have  $H(\mu, \nu) \leq H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}})$ .*

**Proof.** Let  $\lambda_{\mathcal{G}}$  be the restriction of  $\lambda$  to  $(E, \mathcal{G})$ . It is easy to check that

$$\frac{d\mu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_{\lambda} \left( \frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) \quad \frac{d\nu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_{\lambda} \left( \frac{d\nu}{d\lambda} \middle| \mathcal{G} \right), \quad \lambda\text{-a.e.}^{(6)}$$

Consequently we have <sup>(7)</sup>

$$H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}}) = \int_E \left[ E_{\lambda} \left( \frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) E_{\lambda} \left( \frac{d\nu}{d\lambda} \middle| \mathcal{G} \right) \right]^{1/2} d\lambda.$$

Since  $\lambda$ -a.e.

$$\frac{\left[ \frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda} \right]^{1/2}}{\left[ E_{\lambda} \left( \frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) E_{\lambda} \left( \frac{d\nu}{d\lambda} \middle| \mathcal{G} \right) \right]^{1/2}} \leq \frac{1}{2} \left( \frac{\frac{d\mu}{d\lambda}}{E_{\lambda} \left( \frac{d\mu}{d\lambda} \middle| \mathcal{G} \right)} + \frac{\frac{d\nu}{d\lambda}}{E_{\lambda} \left( \frac{d\nu}{d\lambda} \middle| \mathcal{G} \right)} \right),$$

taking conditional expectations of both sides one finds,  $\lambda$ -a.e.,

$$\left[ E_{\lambda} \left( \frac{d\mu}{d\lambda} \middle| \mathcal{G} \right) E_{\lambda} \left( \frac{d\nu}{d\lambda} \middle| \mathcal{G} \right) \right]^{1/2} \geq E_{\lambda} \left( \left( \frac{d\mu}{d\lambda} \right)^{1/2} \left( \frac{d\nu}{d\lambda} \right)^{1/2} \middle| \mathcal{G} \right). \quad (1.3.3)$$

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<sup>6</sup>  $E_{\lambda}(\eta|\mathcal{G})$  is the conditional expectation of the random variable  $\eta$  with respect to  $\mathcal{G}$  and measure  $\lambda$ .

<sup>7</sup> For positive numbers  $a, b, c, d$ ,  $\sqrt{\frac{ab}{cd}} \leq \frac{1}{2} \left( \frac{a}{c} + \frac{b}{d} \right)$ .



Integrating with respect to  $\lambda$  both sides of (1.3.3), the required result follows.  $\square$

Now let us consider two sequences of measures  $(\mu_k)$  and  $(\nu_k)$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that  $\nu_k \sim \mu_k$  for all  $k \in \mathbb{N}$ . We set  $\lambda_k = \frac{1}{2}(\mu_k + \nu_k)$ , and we consider the Hellinger integral

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[ \frac{d\mu_k}{d\lambda_k}(x) \frac{d\nu_k}{d\lambda_k}(x) \right]^{1/2} \lambda_k(dx), \quad k \in \mathbb{N}.$$

**Remark 1.3.4** Since  $(\mu_k)$  and  $(\nu_k)$  are equivalent, we have

$$\frac{d\mu_k}{d\lambda_k} \frac{d\nu_k}{d\lambda_k} = \frac{d\mu_k}{d\lambda_k} \frac{d\nu_k}{d\mu_k} \frac{d\mu_k}{d\lambda_k} = \frac{d\nu_k}{d\mu_k} \left( \frac{d\mu_k}{d\lambda_k} \right)^2.$$

Thus

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[ \frac{d\nu_k}{d\mu_k}(x) \right]^{1/2} \mu_k(dx). \quad (1.3.4)$$

We also consider the product measures on  $\mathbb{R}^\infty$

$$\mu = \prod_{k=1}^{\infty} \mu_k, \quad \nu = \prod_{k=1}^{\infty} \nu_k,$$

and the corresponding Hellinger integral  $H(\mu, \nu)$ . As is easily checked we have

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k).$$

**Proposition 1.3.5 (Kakutani)** *If  $H(\mu, \nu) > 0$  then  $\mu$  and  $\nu$  are equivalent. Moreover*

$$f(x) := \frac{d\nu}{d\mu}(x) = \prod_{k=1}^{\infty} \frac{d\nu_k}{d\mu_k}(x_k), \quad x \in \mathbb{R}^\infty, \quad \mu\text{-a.e.} \quad (1.3.5)$$

**Proof.** We set

$$f_n(x) = \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k), \quad x \in \mathbb{R}^\infty, \quad n \in \mathbb{N}.$$

We are going to prove that the sequence  $(f_n)$  is convergent on  $L^1(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$ . Let  $m, n \in \mathbb{N}$ , then we have

$$\begin{aligned}
& \int_{\mathbb{R}^\infty} \left| f_{n+m}^{1/2}(x) - f_n^{1/2}(x) \right|^2 \mu(dx) \\
&= \int_{\mathbb{R}^\infty} \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k) \left| \prod_{k=n+1}^{n+m} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx) \\
&= \prod_{k=1}^n \int_{\mathbb{R}^\infty} \frac{d\nu_k}{d\mu_k}(x_k) \mu(dx) \int_{\mathbb{R}^\infty} \left| \prod_{k=n+1}^{n+m} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx).
\end{aligned}$$

Consequently

$$\begin{aligned}
& \int_{\mathbb{R}^\infty} |f_{n+p}^{1/2}(x) - f_n^{1/2}(x)|^2 \mu(dx) \\
&= \int_{\mathbb{R}^\infty} \left[ \prod_{k=n+1}^{n+p} \frac{d\nu_k}{d\mu_k}(x_k) - 2 \prod_{k=n+1}^{n+p} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} + 1 \right] \mu(dx) \\
&= 2 \left( 1 - \prod_{k=n+1}^{n+p} \int_{\mathbb{R}} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} \mu_k(dx_k) \right) \\
&= 2 \left( 1 - \prod_{k=n+1}^{n+p} H(\mu_k, \nu_k) \right). \tag{1.3.6}
\end{aligned}$$

On the other hand we know by assumption that

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k) > 0,$$

or, equivalently, that

$$-\log H(\mu, \nu) = -\sum_{k=1}^{\infty} \log[H(\mu_k, \nu_k)] < +\infty.$$

Consequently, for any  $\varepsilon > 0$  there exists  $n_\varepsilon \in \mathbb{N}$  such that if  $n > n_\varepsilon$  and  $p \in \mathbb{N}$ , we have

$$- \sum_{k=n+1}^{n+p} \log[H(\mu_k, \nu_k)] < \varepsilon.$$

By (1.3.6) if  $n > n_\varepsilon$  we have

$$\int_{\mathbb{R}^\infty} |\sqrt{f_{n+p}} - \sqrt{f_n}|^2 d\mu \leq 2(1 - e^{-\varepsilon}).$$

Thus the sequence  $(f_n^{1/2})$  is convergent on  $L^2(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$  to some function  $f^{1/2}$ . Therefore  $f_n \rightarrow f$  in  $L^1(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$ .

Finally, we prove that  $\nu \ll \mu$  and  $f = \frac{d\nu}{d\mu}$ . Let  $\varphi$  be a continuous bounded Borel function on  $\mathbb{R}^\infty$ , and set  $\varphi_n(x) = \varphi(P_n(x))$ ,  $x \in \mathbb{R}^\infty$ , where  $P_n x = \{x_1, \dots, x_n, 0, 0, \dots\}$ . Then we have

$$\begin{aligned} \int_{\mathbb{R}^\infty} \varphi(P_n x) \nu(dx) &= \int_{\mathbb{R}^n} \varphi(P_n x) \nu_1(dx_1) \dots \nu_n(dx_n) \\ &= \int_{\mathbb{R}^n} \varphi(P_n x) \frac{d\nu_1}{d\mu_1}(x_1) \dots \frac{d\nu_n}{d\mu_n}(x_n) \mu_1(dx_1) \dots \mu_n(dx_n) \\ &= \int_{\mathbb{R}^\infty} \varphi(P_n x) f_n(x) \mu(dx). \end{aligned}$$

Letting  $n$  tend to infinity, we find

$$\int_{\mathbb{R}^\infty} \varphi(x) \nu(dx) = \int_{\mathbb{R}^\infty} \varphi(x) f(x) \mu(dx),$$

so that  $\nu \ll \mu$ . Finally, by exchanging the rôles of  $\mu$  and  $\nu$ , we find  $\mu \ll \nu$ .  $\square$

### 1.3.2 The Cameron-Martin formula

We consider here the measures  $\mu = N_{a,Q}$  and  $\nu = N_Q$ , and for any  $a \in Q^{1/2}(H)$  we set

$$\rho_a(x) = \exp \left\{ -\frac{1}{2} |Q^{-1/2} a|^2 + \langle Q^{-1/2} a, Q^{-1/2} x \rangle \right\}, \quad x \in H. \quad (1.3.7)$$

Let us recall, see §1.2.4, that  $W_f(x) = \langle f, Q^{-1/2} x \rangle$  was defined for all  $f \in \overline{Q^{1/2}(H)}$ . Since  $Q^{-1/2} a \in Q^{1/2}(H)$  the definition (1.3.7) is meaningful.